AMS-311. Spring 2005. Homework 2. Topics: Conditional Probability, Bayes' Rule, Independence

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1). A magnetic tape storing information in binary form has been corrupted. Imagine that you are to make an effort to save as much information as possible. Due to the damage on the tape, you know that there will be errors in the reading. You know that if there was a 0, the probability that you correctly detect it is 0.9. The probability that you correctly detect a 1 is 0.85. Given that the each digit is a 1 or a 0 with equal probability, and given that you read in a 1, what is the probability that this is a correct reading?

2). George has lost his dog in either forest A (with a priori probability 0.4) or in forest B (with a priori probability 0.6). If the dog is alive and not found by the n-th day of the search, it will die that evening with probability $\frac{n}{n+2}$.

If the dog is in A (either dead or alive) and George spends a day searching for it in A, the conditional probability that he will find the dog that day is 0.25. Similarly, if the dog is in B and George spends a day looking for it there, he will find the dog that day with probability 0.15.

The dog cannot go from one forest to the other. George can search only in the daytime, and he can travel from one forest to the other only at night.

All parts of this problem are to be worked separately.

(a) In which forest should George look to maximize the probability he finds his dog on the first day of the search?

(b) Given that George looked in A on the first day but did not find his dog, what is the probability that the dog is in A?

(c) If George flips a fair coin to determine where to look on the first day and finds the dog on the first day, what is the probability that he looked in A?

(d) George has decided to look in A for the first two days. What is the a priori probability that he will find a live dog for the first time on the second day?

(e) George has decided to look in A for the first two days. Given the fact that he was unsuccessful on the first day, determine the probability that he does not find a dead dog on the second day.

(f) George finally found his dog on the fourth day of the search. He looked in A for the first 3 days and in B on the fourth day. What is the probability he found his dog alive?

(g) George finally found his dog late on the fourth day of the search. The only other thing we know is that he looked in A for 2 days and in B for 2 days. What is the probability he found his dog alive?

3). **Knives and Forks:** In the kitchen in your apartment, you put all your 10 forks in the left drawer and all 10 knives in the right drawer. Your roommate, who does not agree with your organizational approach, comes in, takes two forks from the left drawer and tosses them into the right drawer. She then takes at random an item (knife or fork) from the right drawer and tosses it in the left drawer. After this exchange, you come in and randomly pick up an item from a randomly chosen drawer. Given you have picked up a knife, what is the probability that you have opened the left drawer?

4). Let $A_1, A_2, \ldots, A_n$ be independent events. Show that

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = 1 - \prod_{i=1}^{n}(1 - P(A_i)).$$

You can use the fact that if $A_1, \ldots, A_n$ are independent, the same is true for $A_1^c, \ldots, A_n^c$.

5). **[Extra Credit]** Suppose that A, B, and C are independent. Use the definition of independence to show that A and $B \cup C$ are independent.

6). **[Extra Credit]** Alice, Bob and Carol play a chess tournament. The first game is played between Alice and Bob. The player who sits out a given game plays next the winner of that game. The tournament ends when some player wins two successive games. Let a tournament history be the list of game winners, so for example ACBAA corresponds to the tournaments where Alice won games 1,4, and 5, Carol won game 2, and Bob won game 3.

(a) Provide a tree-based sequential description of a sample space where the outcomes are the possible tournament histories.

(b) We are told that every possible tournament history that consists of k games has probability $1/2^k$, and that a tournament history consisting of an infinite number of games has zero probability. Demonstrate that this assignment of probabilities defines a legitimate probability law.

(c) Assuming the probability law from part (6b) to be correct, find the probability that the tournament lasts no more than 5 games. Also find the probability that each of Alice, Bob, and Carol wins the tournament.